J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 17–28
Printed in Great Britain

Stability of a visco-elastic liquid film flowing down an inclined plane

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(Received 21 March 1966)

The investigation concerns the stability of an incompressible second-order fluid (visco-elastic) film flowing down an inclined plane under gravity with respect to two-dimensional disturbances. When the elastic parameter is negative as in the case of a solution of polyisobutylene in cetane, surface disturbances ('soft' waves) are found to be unstable. The analysis in this case also reveals the existence of growing shear waves ('hard' waves) which are highly damped in ordinary Newtonian fluids.

1. Introduction

The problem of the stability of the laminar flow of an ordinary viscous liquid film flowing down an inclined plane under gravity was formulated by Yih (1955) who solved the stability equation numerically. Using his formulation, Benjamin (1957) gave an analytical solution for the neutral stability curves for the same problem by a power expansion technique. For a vertical plate, his values for the wave speed were found to be in good agreement with the experimental values of Binnie (1957). Both these investigations reveal the occurrence of instability at small Reynolds numbers. In view of certain inaccuracies in Yih's (1955) results and the extremely laborious nature of Benjamin's (1957) power expansion method, Yih (1963) presented a simple perturbation technique (for the same problem) which furnished information regarding the stability of the flowing film for the cases of small wave-numbers, small Reynolds numbers and of large wave-numbers. He also discussed shear-wave stability for these three cases and showed that when the inclination of the plate with the horizontal is not very small, these waves are strongly damped. Quite recently, Yih (1965) has used the same technique for studying the stability of a non-Newtonian inelastic fluid film whose constitutive equation is triply non-linear. However, his solution for long waves is confined to surface-wave instability only and is valid for small values of the non-Newtonian parameter.

The present investigation is taken up for several reasons. First, we have studied the stability of a film of liquid (flowing under gravity) based on a model of a non-Newtonian liquid which in addition to cross-viscosity displays elastic properties. When a visco-elastic fluid flows, a certain amount of energy is stored up in the material as strain energy in addition to dissipation of heat due to viscosity. Thus it may be expected that stability characteristics of the flow of such a fluid will be influenced by its elastic properties.

Secondly, it may be of some interest to discuss instability of shear waves in such fluids in addition to that of the surface waves. In the sequel, it is shown that for negative values of the elastic parameter the shear waves become unstable.

Thirdly, the present analysis provides a basis for experiments on the stability and other characteristics of visco-elastic liquids. For instance, it is shown that from the critical Reynolds number determined experimentally, the value of the elastic constant in the constitutive equation may be estimated.

The perturbation technique of Yih (1963) will be followed in the present analysis and the result is valid for all values of the elastic parameter.

2. A mathematical model for visco-elastic fluids

Noll (1958) defined an incompressible simple material as a substance whose density remains constant and whose stress is determined, to within an arbitrary hydrostatic pressure, by the history of the strain. This substance is called a simple fluid, if it has the characteristic that all local states are equivalent in response, with all observable differences in response being due to definite differences in history. For any history $g(s)$, a retarded history $g_\alpha(s)$ can be defined as

$$g_\alpha(s) = g(\alpha s) \quad (0 < s < \infty, \quad 0 < \alpha \leqslant 1), \tag{2.1}$$

α being termed as retardation factor. In addition to these properties, assuming that the stress is more sensitive to recent deformations than to deformations in the distant past (postulate of gradually fading memory) Coleman & Noll (1960) showed that the theory of simple fluids leads to that of perfect fluids as $\alpha \to 0$ and that of Newtonian fluids as a correction of order α to the theory of perfect fluids. Based on these ideas, they derived the following equation for an isotropic fluid which includes corrections for visco-elastic effects to $O(\alpha^2)$

$$S_{ij} + p\delta_{ij} = \eta_0 A_{(1)ij} + \beta A_{(1)ik} A_{(1)kj} + \gamma A_{(2)ij}, \tag{2.2}$$

where S_{ij} is the stress tensor, p is an indeterminate pressure (and no longer the mean pressure) and η_0, β and γ are material constants. The tensors $A_{(N)ij}$ known as Rivlin–Ericksen tensors (1955) are connected with the rate-of-strain tensor $v_{i,j}$ by the following recursion formulae:

$$A_{(1)ij} = v_{i,j} + v_{j,i}, \tag{2.3}$$

$$A_{(N)ij} = A_{(N-1)ik} v_{k,j} + A_{(N-1)jk} v_{k,i} + \dot{A}_{(N-1)ij}, \tag{2.4}$$

where the over dot denotes the material time derivative. In (2.2), the term containing β arises out of cross-viscosity while the last term accounts for the elastic properties of the fluid. Using (2.3) and (2.4), the acceleration gradient term $A_{(2)ij}$ in (2.2) is given by

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i} v_{m,j}, \tag{2.5}$$

where a_i are the components of acceleration and are given by $\partial v_i/\partial t + v_j v_{i,j}$.

It has been reported by Markovitz & Coleman (1964) that a 5·4 % solution of polyisobutylene in cetane at 30 °C obeys the relation (2.2) and Markovitz &

Brown (see Markovitz & Coleman 1964) found the following values from normal stress measurements for this fluid

$$\eta_0 = 18{\cdot}5\,\mathrm{P}, \quad \rho = 0{\cdot}773\,\mathrm{g/cm^3}, \quad \beta = 1{\cdot}0\,\mathrm{g/cm}, \quad \gamma = -0{\cdot}2\,\mathrm{g/cm}. \tag{2.6}$$

The constitutive equation (2.2) will be used in our analysis. Of course, this equation will be valid only if the shearing rates are not too large.

3. Mathematical formulation and the stability analysis

A layer of a visco-elastic liquid of thickness d flows down a plane (figure 1) inclined at an angle β_0 to the horizon. The steady primary flow is taken parallel to the x_1-axis with the x_2-axis normal to the plate downwards, the origin being taken at the undisturbed free surface. The x_3-axis normal to the x_1- and x_2-axes is not shown in the figure.

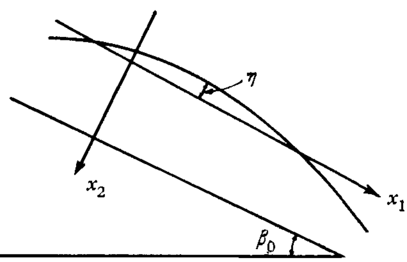

Fig. 1. A sketch of the physical problem.

The equations of momentum and continuity are

$$\rho(\partial v_i/\partial t + v_j\,\partial v_i/\partial x_j) = \partial S_{ij}/\partial x_j + \rho X_i, \tag{3.1}$$

$$\partial v_j/\partial x_j = 0, \tag{3.2}$$

where X_i are the components of force due to gravity and S_{ij} is given by (2.2).

The primary flow is steady and unidirectional and the velocity depends on x_2 only. Using the superscript 0 to denote various quantities for this flow, (3.1) gives

$$-\frac{\partial}{\partial x_1}\left[p^0 - (\beta+2\gamma)\left(\frac{dv_1^0}{dx_2}\right)^2\right] + \eta\frac{d^2v_1^0}{dx_2^2} + \rho g\sin\beta_0 = 0, \tag{3.3}$$

$$-\frac{\partial}{\partial x_2}\left[p^0 - (\beta+2\gamma)\left(\frac{dv_1^0}{dx_2}\right)^2\right] + \rho g\cos\beta_0 = 0, \tag{3.4}$$

$$\frac{\partial}{\partial x_3}\left[p^0 - (\beta+2\gamma)\left(\frac{dv_1^0}{dx_2}\right)^2\right] = 0, \tag{3.5}$$

where use is made of (2.2), (2.3) and (2.5). From the above equations it follows that

$$\frac{\partial}{\partial x_1}\left[p^0 - (\beta+2\gamma)\left(\frac{dv_1^0}{dx_2}\right)^2\right] = K, \quad \text{a constant.} \tag{3.6}$$

Since $S_{22} = 0$ at the free surface, (2.2) gives

$$-p^0 + (\beta+2\gamma)\,(dv_1^0/dx_2)^2 = 0$$

at the free surface so that K in (3.6) is zero. The velocity distribution is now obtained from (3·3) as

$$v_1^0(x_2) = \rho g\sin\beta_0(d^2 - x_2^2)/2\eta, \tag{3.7}$$

satisfying the no-slip condition $v_1^0 = 0$ at $x_2 = d$ and the stress free condition $S_{12} = 0$ at the free surface $x_2 = 0$. It is of some interest to note that the primary velocity is affected neither by cross-viscosity nor by elasticity. These effects manifest themselves only in modifying the pressure distribution.

In our stability analysis we shall assume the validity of Squire's theorem, viz. the two-dimensional disturbances are more unstable than the three-dimensional ones. This enables us to restrict consideration to two-dimensional perturbations only. In fact, Binnie's (1957) experiments with vertical water films showed that the waves at small Reynolds numbers were approximately uniform along the horizontal line of their crests. This two-dimensional nature of surface waves (roll waves) may therefore be expected to hold good for visco-elastic liquids also.

We take the physical variables in the perturbed state as

$$v_1 = v_1^0 + u_1, \quad v_2 = u_2, \quad p = p^0 + P. \tag{3.8}$$

Substituting these in (3.1), (3.2) and using (2.2) and (2.5), we obtain after linearization, the following equations

$$\begin{aligned}\rho\left(\frac{\partial u_1}{\partial t}+v_1^0\frac{\partial u_1}{\partial x_1}+u_2\frac{dv_1^0}{dx_2}\right) &= -\frac{\partial P}{\partial x_1}+\frac{\partial}{\partial x_1}\left[2\eta_0\frac{\partial u_1}{\partial x_1}+2\beta\frac{dv_1^0}{dx_2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+2\gamma\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t}+v_1^0\frac{\partial u_1}{\partial x_1}+u_2\frac{dv_1^0}{dx_2}\right)\right]\\ &\quad+\frac{\partial}{\partial x_2}\left[\left(\eta_0+\gamma\frac{\partial}{\partial t}+\gamma v_1^0\frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+\gamma u_2\frac{d^2v_1^0}{dx_2^2}+2\gamma\frac{\partial u_1}{\partial x_1}\frac{dv_1^0}{dx_2}\right],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\rho\left(\frac{\partial u_2}{\partial t}+v_1^0\frac{\partial u_2}{\partial x_1}\right) &= -\frac{\partial P}{\partial x_2}+\frac{\partial}{\partial x_1}\left[\left(\eta_0+\gamma\frac{\partial}{\partial t}+\gamma v_1^0\frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)+\gamma u_2\frac{d^2v_1^0}{dx_2^2}+2\gamma\frac{\partial u_1}{\partial x_1}\frac{dv_1^0}{dx_2}\right]\\ &\quad+\frac{\partial}{\partial x_2}\left[2\eta_0\frac{\partial u_2}{\partial x_2}+2\beta\frac{dv_1^0}{dx_2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{dx_1}\right)\right.\\ &\quad\left.+2\gamma\left\{\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial t}+v_1^0\frac{\partial u_2}{\partial x_1}\right)+2\frac{dv_1^0}{dx_2}\frac{\partial u_1}{\partial x_2}\right\}\right],\end{aligned} \tag{3.10}$$

$$\partial u_1/\partial x_1 + \partial u_2/\partial x_2 = 0. \tag{3.11}$$

Introducing the dimensionless variables

$$\left.\begin{aligned} u &= u_1\eta_0/d^2\rho g\sin\beta_0, & v &= u_2\eta_0/d^2\rho g\sin\beta_0,\\ \tau &= td\rho g\sin\beta_0/\eta_0, & x &= x_1/d, y = x_2/d,\end{aligned}\right\} \tag{3.12}$$

and eliminating P from (3.9) and (3.10), the following equation is obtained after using (3.7)

$$\begin{aligned}&\left\{\frac{\partial}{\partial\tau}+\tfrac{1}{2}(1-y^2)\frac{\partial}{\partial x}\right\}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)-v\\ &\quad= 4M\frac{\partial^2}{\partial x\,\partial y}\left[\left\{\frac{1}{MR}+\frac{\partial}{\partial\tau}+\tfrac{1}{2}(1-y^2)\frac{\partial}{\partial x}\right\}\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]+M\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)\\ &\qquad\times\left[\left\{\frac{1}{MR}+\frac{\partial}{\partial\tau}+\tfrac{1}{2}(1-y^2)\frac{\partial}{\partial x}\right\}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2y\frac{\partial u}{\partial x}-v\right],\end{aligned} \tag{3.13}$$

where $$M = \gamma/d^2\rho, \quad R = d^3\rho^2 g \sin\beta_0/\eta_0^2. \tag{3.14}$$

Further, (3.11) reduces to

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{3.15}$$

which permits the use of a stream function ψ defined by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x. \tag{3.16}$$

Assuming $$\psi = \phi(y)\, e^{i\alpha(x-c\tau)} \tag{3.17}$$

and substituting in (3.13) the following equation is derived

$$\begin{aligned}4[(-\alpha^2 + i\alpha^3 cMR)\,\phi'' - \tfrac{1}{2}i\alpha^3 MR\, d\{(1-y^2)\,\phi'\}/dy + i\alpha RM\, d(y\phi'')/dy]\\ + [(1-i\alpha cMR)\,(\phi^{\text{iv}} + 2\alpha^2\phi'' + \alpha^4\phi) + \tfrac{1}{2}i\alpha RM\, d^2\{(1-y^2)\,(\phi'' + \alpha^2\phi)\}/dy^2\\ + \tfrac{1}{2}i\alpha^3 RM(1-y^2)\,(\phi'' + \alpha^2\phi) - 2i\alpha RM\, d^2\{y\phi'(y)\}/dy^2\\ - 2i\alpha^3 RMy\phi'(y) + i\alpha RM(\phi'' + \alpha^2\phi)]\\ = i\alpha R[\{\tfrac{1}{2}(1-y^2) - c\}\,(\phi'' - \alpha^2\phi) + \phi].\end{aligned} \tag{3.18}$$

This is the Orr–Sommerfeld equation modified to take account of elastic properties of the fluid. The boundary conditions at the inclined plane are

$$u = v = 0 \quad \text{at} \quad y = 1,$$

and in terms of ϕ these conditions are

$$\phi'(1) = 0, \quad \phi(1) = 0. \tag{3.19}$$

The conditions at the free surface are complicated since they are to be applied at the perturbed surface rather than at $y = 0$. Let $\eta = \xi d$ be the (dimensional) displacement of the free surface from its mean position so that the kinematic condition at the free surface is

$$\partial\eta/\partial t + v_1^0(0)\,\partial\eta/\partial x_1 = u_2.$$

In terms of non-dimensional variables, the above equation can be written as

$$\partial\xi/\partial\tau + \tfrac{1}{2}(\partial\xi/\partial x) = v$$

at the free surface. Assuming $\xi \sim e^{i\alpha(x-c\tau)}$ and using (3.16) and (3.17), the foregoing equation gives

$$\xi = [\phi(0)/c']\, e^{i\alpha(x-c\tau)}, \quad c' = c - \tfrac{1}{2}. \tag{3.20}$$

Again at the free surface the shear stress vanishes and the normal stress just balances that due to surface tension.

Using (2.2) and (2.5), the shear stress at the free surface due to the perturbation motion is given by

$$S'_{12} = \left(\eta_0 + \gamma\frac{\partial}{\partial t} + \gamma v_1^0\frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \gamma u_2\frac{d^2 v_1^0}{\partial x_2^2} + 2\gamma\frac{dv_1^0}{dx_2}\frac{\partial u_1}{\partial x_1} - \rho g d\xi \sin\beta_0, \tag{3.21}$$

where the last term gives the effect of the variation in the mean shear due to the deviation of the free surface from its mean position. For applying the boundary condition $S'_{12} = 0$ at the free surface, it will be only necessary to evaluate the first three terms on the right-hand side of (3.21) at $y = 0$. Upon using (3.16), (3.17) and (3.20), this boundary condition in a dimensionless form reduces to

$$(1 - Ri\,\alpha c' M)\{\phi''(0) + \alpha^2 \phi(0)\} + (i\alpha MR - 1/c')\,\phi(0) = 0. \qquad (3.22)$$

Again using (2.2), (2.5) and linearizing, we have the following condition on the normal stress at the free surface

$$-\frac{\partial}{\partial x_2}\left[p^0 - (\beta + 2\gamma)\left(\frac{dv_1^0}{dx_2}\right)^2\right]\eta - P + 2\eta_0\frac{du_2}{dx_2} + 2\beta\frac{\partial v_1^0}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$
$$+ 2\gamma\left[\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial t} + v_1^0\frac{\partial u_2}{\partial x_1}\right) + 2\frac{dv_1^0}{dx_2}\frac{\partial u_1}{\partial x_2}\right] + T\frac{\partial^2 \eta}{\partial x^2} = 0, \qquad (3.23)$$

where T is the surface tension (assumed constant) and P is the perturbation pressure. In this equation, apart from the terms involving η, all the other terms are to be evaluated at $y = 0$. Assuming the (x, τ) dependence of the dimensionless pressure $P\eta_0^2/\rho^3 g d^4$ as $e^{i\alpha(x - c\tau)}$ (cf. equation (3.17)), P can be eliminated from (3.9) and (3.23). Using (3.12), (3.16) and (3.17), the result of this elimination can be put in the following dimensionless form

$$(1 - RMi\alpha c')\,\phi'''(0) + [3iM\alpha^3 c' R - 2i\alpha MR + i\alpha c' R - 3\alpha^2]\,\phi'(0)$$
$$+ [i\alpha\cot\beta_0/c' + i\alpha^3 SR/c' + i\alpha RM]\,\phi(0) = 0, \quad (3.24)$$

where $S = T\eta_0^2/\rho^3 g^2 d^5 \sin^2\beta_0$. This is the final form of the boundary condition on the normal stress. Equation (3.18) and the boundary conditions (3.19), (3.22) and (3.24) constitute an eigenvalue problem. For a non-trivial solution a relation

$$c = c(R, M, \alpha, S)$$

must hold good among R, M, S, α and C. From this relation, the curves of neutral stability given by $c_i = 0$ (where $c = c_r + ic_i$) can be plotted.

4. Solution for long waves

We consider disturbances of wavelengths large compared with the depth d. In this case α is small and a solution of the differential system can be found by successive approximation with α as a small parameter. It can be seen from (3.18) that on putting α (or R) equal to zero, the order (fourth) of the equation is not diminished for any M. The eigenfunction is also an entire function of α, R and M as can be seen by solving (3.18) in a power series and considering its convergence for finite values of α, R and M. Further, the boundary conditions are also not singular in α, R or M if they are finite. Thus unlike the singular perturbation problem associated with the Orr–Sommerfeld equation for very large R, we have here a regular perturbation problem for small α (or small R) for which a uniformly valid solution can be found.

Thus, for the first approximation ϕ_1 (with eigenvalue $c_1' = c_1 - \frac{1}{2}$) we put $\alpha = 0$ in (3.18) and the boundary conditions. Hence

$$\phi_1^{\mathrm{iv}}(y) = 0 \tag{4.1}$$

subject to

$$\phi_1'(1) = 0, \quad \phi_1(1) = 0, \quad \phi_1''(0) = \phi_1(0)/c_1', \quad \phi_1'''(0) = 0. \tag{4.2}$$

These give
$$\phi_1(y) = (1-y)^2, \quad c_1' = \tfrac{1}{2} \quad \text{or} \quad c_1 = 1, \tag{4.3}$$

where the arbitrary constant in $\phi_1(y)$ is taken to be unity without loss of generality. This shows that $\alpha = 0$ is a part of the neutral stability curve. To see how the eigenvalue c is modified as α increases from zero, we consider the second approximation ϕ_2. Correct to $O(\alpha)$, this equation can be obtained from (3.18) as

$$\phi_2^{\mathrm{iv}} + \tfrac{1}{2}i\alpha MR\, d^2\{(1-y^2)\,\phi_1''\}/dy^2 - 2Mi\alpha R d^2\{y\phi_1'(y)\}/dy^2 + i\alpha MR\phi_1'' + 4Mi\alpha R\, d(y\phi_1'')/dy = i\alpha R[\{\tfrac{1}{2}(1-y^2) - c_1\}\,\phi_1'' + \phi_1], \tag{4.4}$$

which upon substitution from (4.3) becomes

$$\phi_2^{\mathrm{iv}} = -2i\alpha Ry. \tag{4.5}$$

Along with the no-slip conditions $\phi_2(1) = 0$ and $\phi_2'(1) = 0$, ϕ_2 has to satisfy (3.22) and (3.24) which, correct to $O(\alpha)$, can be written as

$$\phi_2''(0) + [\tfrac{1}{2}MRi\alpha - MRi\alpha c_1]\,\phi_1''(0) + i\alpha MR\phi_1(0) - \phi_2(0)/c_1' + \Delta c\phi_1(0)/c_1'^2 = 0 \tag{4.6}$$

and

$$\phi_2'''(0) = 2MRi\alpha\phi_1'(0) - MRi\alpha\phi_1(0) - Ri\alpha c_1'\,\phi_1'(0) - i\alpha\cot\beta_0\,\phi_1(0)/c_1'. \tag{4.7}$$

In (4.6), Δc stands for the change in c_1' as α deviates from zero. It may be noted that although the first-order approximation ϕ_1 given by (4.3) does not contain the elastic parameter M, the second approximation ϕ_2 includes M through the boundary conditions (4.6) and (4.7). Solution of (4.5) subject to the aforementioned boundary conditions gives

$$\Delta c = i\alpha(\tfrac{2}{15}R - \tfrac{1}{3}\cot\beta_0 - \tfrac{5}{6}RM). \tag{4.8}$$

This shows that while $c_i = 0$ at $\alpha = 0$, c_i will increase or decrease as α increases from zero, according as

$$R \gtrless 10\cot\beta_0/(4-25M). \tag{4.9}$$

Thus the neutral stability curve in the (α, R)-plane has a bifurcation point at $\alpha = 0$, $R = R_c$ where R_c is given by

$$R_c = 10\cot\beta_0/(4-25M), \tag{4.10}$$

which agrees with Yih's (1963) result for $M = 0$. From (2.6) and (3.14) it is clear that for a solution of polyisobutylene in cetane $M < 0$, which shows that the critical Reynolds number is less than the corresponding value for an ordinary viscous fluid. Using (2.6) and (4.10), the critical Reynolds number, for a film of thickness 2 cm of this fluid flowing down a plane with $\beta_0 = \frac{1}{4}\pi$, is roughly 1·778 while for an ordinary viscous fluid this value is 2·5. It is also clear from (4.10) that the bifurcation point shifts towards the origin in the (α, R)-plane as $|M|$ increases and R_c becomes zero at $\beta_0 = \frac{1}{2}\pi$ (vertical plate). This shows that for such a fluid, the elastic effects are destabilizing. The opposite will be true for a fluid with $M > 0$. However, from thermodynamic consideration it can be shown that $M < 0$ as discussed by Markovitz & Coleman (1964). It may be pointed out that (4.10) serves as a basis for estimating M (and therefore γ) for a given inclination β_0 when R_c is experimentally determined.

We can also discuss the stability characteristics for small R. This will also be a regular perturbation problem since the order of (3.18) is not diminished by putting $R = 0$. In this case, the first approximation ϕ_1 will satisfy

$$\phi_1^{\text{iv}} - 2\alpha^2\phi_1'' + \alpha^4\phi_1 = 0 \tag{4.11}$$

subject to
$$\phi_1'(1) = 0, \quad \phi_1(1) = 0, \quad \phi_1''(0) + (\alpha^2 - 1/c_1')\,\phi_1(0) = 0 \tag{4.12}$$

and
$$-(i\alpha/c_1')(\cot\beta_0 + R\alpha^2 S)\,\phi_1(0) + 3\alpha^2\phi_1'(0) - \phi_1'''(0) = 0, \tag{4.13}$$

where SR is independent of viscosity η_0. The solution of (4.11) is

$$\phi_1 = L_1 e^{\alpha y} + M_1 e^{-\alpha y} + N_1 y e^{\alpha y} + P_1 y e^{-\alpha y}. \tag{4.14}$$

This gives after using (4.12) and (4.13)

$$c_1' = [1 + i(2\alpha - \sinh 2\alpha)(\cot\beta_0 + R\alpha^2 S)/2\alpha^2]/(1 + \cosh 2\alpha + 2\alpha^2).$$

This shows that for $\beta_0 = \frac{1}{2}\pi$ and $S = 0$, c_1', is real so that $R = 0$ is a part of the neutral stability curve even in a visco-elastic liquid. The higher-order approximations will be affected by M and can be carried out exactly as before without any difficulty. From a physical point of view, the instability of a vertical film at all R (including $R = 0$) arises from the fact that in the absence of the stabilising influence of $\beta_0 \neq \frac{1}{2}\pi$ and surface tension, disturbances grow at the expense of the strain energy of the elastic elements as well as of the energy of the gravity field.

5. Shear wave instability

So far we have restricted our attention to surface waves only as distinct from shear waves which occur in confined flows at large values of R. These waves (surface waves) are found to be unstable for small Reynolds numbers. Elastic properties of the fluid tend to make these waves even more unstable. It will be of some interest to study the stability of shear waves in such fluids. In ordinary viscous fluids, however, these waves have been shown to be highly damped for small Reynolds numbers or small wave-numbers and stability characteristics are governed by surface waves as shown by Yih (1963).

We first consider the stability of Poiseuille flow of a visco-elastic liquid between two parallel plates for the following two cases: (1) small Reynolds number for any α; (2) small wave number for any finite R. As the flow of the liquid down an inclined plane is one half of a plane Poiseuille flow, it will be of interest to see whether the shear waves are amplified or damped in a plane Poiseuille flow.

Case 1. Here we assume $R \ll 1$ but Rc is not small. In this case, (3.18) reduces to

$$(D^2-\alpha^2)(D^2-\beta_1^2)\phi = 0, \tag{5.1}$$

where $D = d/dy$ and β_1 is given by

$$\beta_1^2 = \alpha^2 + i\alpha Rc/(Mi\alpha Rc - 1). \tag{5.2}$$

With the plates at $y = \pm 1$, the boundary conditions are

$$\phi(\pm 1) = 0, \quad \phi'(\pm 1) = 0. \tag{5.3}$$

The assumption that Rc is not small is necessary otherwise c will completely drop out of the differential system and we shall not have a non-trivial solution.

The even solution (antisymmetric mode) of (5.1) is

$$\phi = A \cosh \alpha y + B \cosh \beta_1 y, \tag{5.4}$$

which with (5.3) leads to

$$\beta_1 \tanh \beta_1 = \alpha \tanh \alpha.$$

Putting $\beta_1 = \gamma_1 i$, the above equation becomes

$$\gamma_1 \tan \gamma_1 = -\alpha \tanh \alpha. \tag{5.5}$$

This equation has an infinite number of real roots γ_{1n} $(n = 1, 2, \ldots)$ at a distance π apart asymptotically so that, from (5.2), the corresponding eigenvalues c_n are given by

$$Rc_n = -i(\gamma_{1n}^2+\alpha^2)/\alpha[1+M(\gamma_{1n}^2+\alpha^2)]. \tag{5.6}$$

Since $M < 0$, for wave-numbers satisfying $|M|(\gamma_{1n}^2+\alpha^2) > 1$ the coefficient of i on the right-hand side of (5.6) is positive. Thus even at small Reynolds numbers, the shear waves are amplified. The assumption that Rc is not small is, of course, consistent with (5.6).

Case 2. Here we assume $\alpha \ll 1$ but αc is not small. The governing equation is

$$\phi^{\text{iv}} - b^2\phi'' = 0, \tag{5.7}$$

where

$$b^2 = -i\alpha cR/(1-iMR\alpha c). \tag{5.8}$$

The even solution of (5.7) can be taken as

$$\phi(y) = A_1 + B_1 \cosh by,$$

which with the boundary conditions $\phi'(\pm 1) = 0$ leads to $\sinh b = 0$, i.e. $b = n\pi i$. Hence from (5.8), the eigenvalues c_n are given by

$$c_n = -in^2\pi^2/\alpha R(Mn^2\pi^2+1), \tag{5.9}$$

and for $|M|n^2\pi^2 > 1 (M < 0)$, the coefficient of i in the above equation becomes positive, again leading to instability. The odd solution of (5.7) subject to $\phi(\pm 1) = 0$ will also lead to similar results. However, the disturbances corresponding to the even solution can be shown to be more unstable than those due to the odd solution.

Thus in plane Poiseuille flows, shear waves are amplified when $M < 0$ even at small R. It is, therefore, expected that these waves are also unstable in a film flow. However, in view of the complex nature of the boundary conditions at the free surface, we shall use an integral method for the case of small R.

When $R \ll 1$ but Rc is not small, the governing differential equation is (5.1), which can be written as

$$(D^2-\alpha^2)^2\phi = [i\alpha Rc/(Mi\alpha Rc-1)](D^2-\alpha^2)\phi, \tag{5.10}$$

subject to the no-slip conditions (3.19) and the normal-stress condition (3.24) at the free surface. Letting $R \to 0$ and c (or c') $\to \infty$ (since Rc is finite), the free-boundary condition (3.22) for the tangential stress becomes

$$\phi''(0)+\alpha^2\phi(0) = 0. \tag{5.11}$$

Multiplying (5.10) by ϕ^* (complex conjugate of ϕ) and integrating between 0 and 1, we obtain, after using (3.19),

$$-\phi^*(0)\phi'''(0)+\phi'^*(0)\phi''(0)+I_2+2\alpha^2\phi^*(0)\phi'(0)+2\alpha^2I_1+\alpha^4I_0$$
$$= [i\alpha Rc/(1-Mi\alpha Rc)][\phi^*(0)\phi'(0)+I_1+\alpha^2I_0], \tag{5.12}$$

where $$I_0 = \int_0^1 |\phi|^2dy, \quad I_1 = \int_0^1 |\phi'|^2dy, \quad I_2 = \int_0^1 |\phi''|^2dy. \tag{5.13}$$

Again letting $R \to 0$ and $c' \to \infty$ in (3.24), we obtain

$$\phi'''(0)-3\alpha^2\phi'(0) = i\alpha Rc\phi'(0)/(Mi\alpha Rc-1). \tag{5.14}$$

Elimination of $\phi'''(0)$ between (5.12) and (5.14) and subsequent use of (5.11) lead to

$$I_2+2\alpha^2I_1+\alpha^4I_0-\alpha^2[\phi(0)\phi'^*(0)+\phi^*(0)\phi'(0)]$$
$$= [i\alpha Rc/(1-Mi\alpha Rc])[I_1+\alpha^2I_0]. \tag{5.15}$$

Since $$\phi(0)\phi'^*(0)+\phi^*(0)\phi'(0) = 2I_1+\int_0^1[\phi''^*\phi+\phi''\phi^*]dy,$$
we can write (5.15) as

$$\frac{i\alpha Rc(I_1+\alpha^2I_0)}{1-Mi\alpha Rc} = I_2+\alpha^4I_0-\alpha^2\int_0^1[\phi''^*\phi+\phi''\phi^*]dy$$
$$= \int_0^1|\phi''-\alpha^2\phi|^2dy. \tag{5.16}$$

The right-hand side of (5.16) is positive definite, for otherwise

$$\phi''-\alpha^2\phi = 0$$

and its solution $$\phi = A_2\cosh\alpha y+B_2\sinh\alpha y$$

cannot satisfy $\phi(1) = \phi'(1) = 0$ non-trivially.

Letting $c = c_r+ic_i$ and equating real and imaginary parts of (5.16), we find that $c_r = 0$ and

$$\frac{-c_i\alpha R(I_1+\alpha^2I_0)}{(1+M\alpha Rc_i)} = \int_0^1|\phi''-\alpha^2\phi|^2dy. \tag{5.17}$$

This means that for any $\alpha\,(>0)$

$$c_i(M\alpha Rc_i+1)^{-1} < 0. \tag{5.18}$$

For $M = 0$, (5.18) clearly implies $c_i < 0$ so that shear-wave disturbances are damped in ordinary viscous liquids (cf. Yih (1963)). However, for a visco-elastic liquid $M < 0$ and (5.18) is valid with $c_i > 0$ for a suitable range of wave-numbers. This implies the instability of shear-wave disturbances.

For small wave-numbers, we can obtain more definite results. In this case we assume that although $\alpha \ll 1$, αc is not small. The governing differential equation in this case is (5.7) subject to the no-slip conditions (3.19) while the other two boundary conditions become

$$\phi''(0) = 0, \quad b^2\phi'(0) - \phi'''(0) = 0, \tag{5.19}$$

where b is defined by (5.8).

The solution of (5.7) can be written as

$$\phi(y) = A_3 + B_3 y + C_3 e^{by} + D_3 e^{-by}, \tag{5.20}$$

which gives after using the above boundary conditions

$$A_3 = 2D_3 e^b, \quad B_3 = 0, \quad C_3 = -D_3, \quad e^b + e^{-b} = 0. \tag{5.21}$$

The last condition in (5.21) implies $2b = (2n+1)\,\pi i$ and this gives from (5.8)

$$4i\alpha Rc = (1 - MRi\alpha c)\,(2n+1)^2\pi^2. \tag{5.22}$$

Equating its real part,

$$-Rc_i\alpha[M(2n+1)^2\pi^2 + 4] = (2n+1)^2\pi^2. \tag{5.23}$$

For $|M|\,(2n+1)^2\pi^2 > 4$ and since $M < 0$, this shows that $(c_i > 0)$, leading to unstable shear waves whose growth rates are αc_i. From (5.23), the assumption that αc is not small is justified *a posteriori*.

6. Discussion

The present investigation shows that, for an incompressible second-order fluid flowing down an inclined plane, surface wave instability sets in at a critical Reynolds number smaller than the corresponding value in an ordinary Newtonian fluid. Thus the second-order effects are destabilising. On the other hand, the shear waves become unstable at any Reynolds number. Thus in film flow of such fluids the instability is governed by shear waves rather than surface waves and this is contrary to the corresponding result in an ordinary viscous fluid. Physically this instability may be explained as follows. When the flow of such a fluid (visco-elastic) down a plane is subject to a disturbance, the shear on an element of the fluid is reversed at such a frequency that the elastic stresses cannot relax. This results in a decrease in dissipation of the energy of the disturbance, part of the energy being stored in the element as strain energy. Thus we may expect instability in flows of such fluids. However, it should be noted that the above-mentioned elastico-viscous properties refer to the bulk of the fluid and are distinct from those of a free surface of a fluid in which some contaminants or surface active agents (surfactants) are present. These surfactants ascribe to the surface both viscous and elastic properties. In the absence of mass transfer between the surface and the bulk of the fluid, the surface tension is connected with the dilatational deformation and this explains surface elasticity. It has been shown by Levich (1962) and Berg & Acrivos (1965) that this surface elasticity is stabilizing because it tends to suppress wave formation contrary to our results.

Finally, we make a few remarks about the suitability of the model of the second-order fluid as representing the actual behaviour of elastico-viscous fluids. A serious shortcoming of this model is that it fails to exhibit gradual stress relaxation generally observed in elastico-viscous fluids despite the fact that it accounts for shear dependent viscosity and normal stress effects. When the tensors $A_{(N)ij}$ $(N = 1, 2)$ (defined earlier) in the constitutive equation of this fluid vanish, it is clear that the stress cannot change in time but tends to the hydrostatic pressure precipitously. It is indeed true that second-order fluids exhibit retarded response to applied stresses, but the normal-stress differences in such fluids do not correspond to a simple tension along the streamlines. Despite these shortcomings, Truesdell (1965) has recently given some rationalization for using the model of the second-order fluid as representing the behaviour of an elastico-viscous fluid. He has defined a fluid with 'convected elasticity' as one in which the stress is a function of the strain of the present configuration with respect to one occupied by the fluid at a certain fixed time t^* before the present time, t^* being called the response time. He has further shown that in viscometric or shear flows, a second-order fluid is indistinguishable from a fluid with 'convected elasticity' with response time $t^* = -2\gamma/\eta_0$. Hence in order that t^* may be positive (since stress depends on the past and not on the future) γ must be negative. Markovitz & Coleman (1964), however, have established the negativeness of γ from thermodynamic considerations. In the present problem we have studied the stability of a basic shear flow. Hence from the foregoing considerations of Truesdell we may expect that the second-order fluid model will be a reasonably good approximation to an elastico-viscous fluid. Further we can anticipate on physical grounds that elastic properties of a fluid are brought out more clearly in an unsteady flow (such as the unsteady perturbed two-dimensional flow in the present stability analysis) than in a steady flow.

The basic nature of the instability of second-order fluids has also been discussed by Coleman, Duffin & Mizel (1965). It is not yet known whether the growth of linearized disturbances will be limited by a breakdown in the fluid model, approximation or by other non-linear effects.

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